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Multiple fixed-sign solutions for a system of higher order three-point boundary-value problems with deviating arguments

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Abstract

We consider the following *system* of higher order three-point boundary-value problems

$$u_i^{(m_i)}(t) = f_i(t, u_1(\phi_1(t)), u_2(\phi_2(t)), \ldots, u_n(\phi_n(t))), \quad t \in [a, b]$$
$$u_i^{(j)}(a) = 0, \quad 0 \le j \le m_i - 3$$
$$u_i^{(m_i-2)}(t^*) = 0, \qquad \xi u_i^{(m_i-3)}(b) + \delta u_i^{(m_i-1)}(b) = 0$$

where $i = 1, 2, \ldots, n$, $m_i \ge 3$, $\frac{1}{2}(a+b) < t^* < b$, $\xi \ge 0$, $\delta > 0$ and ϕ_i's are deviating arguments. Several criteria are offered for the existence of three *fixed-sign* solutions $(u_1, u_2, \ldots, u_n)$ of the system, i.e. for each $1 \le i \le n$ and all $t \in [a, b]$, $\theta_i u_i(t) \ge 0$ where $\theta_i \in \{1, -1\}$ is predetermined. We also present some examples to illustrate the results obtained.

Keywords: Fixed-sign solutions; System of higher order three-point boundary-value problems; Deviating arguments

1. Introduction

In this paper we shall consider a *system* of higher order three-point boundary-value problems

$$\begin{cases} u_i^{(m_i)}(t) = f_i(t, u_1(\phi_1(t)), u_2(\phi_2(t)), \ldots, u_n(\phi_n(t))), & t \in [a, b] \\ u_i^{(j)}(a) = 0, \quad 0 \le j \le m_i - 3 \\ u_i^{(m_i-2)}(t^*) = 0, \qquad \xi u_i^{(m_i-3)}(b) + \delta u_i^{(m_i-1)}(b) = 0 \end{cases} \tag{A}$$

where $i = 1, 2, \ldots, n$, $m_i \ge 3$, ϕ_i's are deviating arguments, t^*, ξ and δ are constants satisfying

$$\frac{1}{2}(a+b) < t^* < b, \qquad \xi \ge 0, \qquad \delta > 0, \qquad \eta \equiv 2\delta + \xi(b-a)(b+a-2t^*) > 0.$$

The importance of boundary value problems, both from a theoretical perspective as well as for their applications in the physical and engineering sciences, has been well documented in the literature; see for example [1–3] and the

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references cited therein. A special case of the system (A) when $n = 1, m_1 = 3, \phi_1(t) = t, \xi = 0, \delta = 1$, i.e.,

$$y'''(t) = f(t, y(t)), \quad t \in [0, 1]; \qquad y(0) = y'(t^*) = y''(1) = 0$$

has been studied by Anderson and Davis [4]. A more general problem, which is another particular case of (A) when $n = 1, m_1 = 3, \phi_1(t) = t$, has also been discussed by Anderson in [5] where the Green's function is developed. Very recently, Graef, Henderson and Yang [6] have tackled an eigenvalue problem, which is also a special case of (A) when $n = 1, \phi_1(t) = t, \xi = 0, \delta = 1$, i.e.,

$$y^{(m)}(t) = \lambda g(t) f(y(t)), \quad t \in [0, 1]; \qquad y^{(j)}(0) = 0, \quad 0 \le j \le m-3, \qquad y^{(m-2)}(t^*) = y^{(m-1)}(1) = 0,$$

where the ranges of λ are established for the existence of a positive solution. Work in other directions (existence of one or two solutions, eigenvalue problem) for another particular case of (A) when $m_i = 3, \phi_i(t) = t, 1 \le i \le n$ can be found in [7,8]. Other multipoint boundary value problems have also attracted a lot of attention [9–12].

Motivated by the research done on multipoint boundary value problems, we shall consider the system (A) which generalizes the problems considered in [4–8] to, firstly, a *system* of *higher order* boundary value problems, and secondly, with very *general* nonlinear terms f_i involving *deviating arguments* — this yields a much more robust model for many nonlinear phenomena.

A solution $u = (u_1, u_2, \ldots, u_n)$ of (A) will be sought in $C^{(m_1)}[a, b] \times C^{(m_2)}[a, b] \times \cdots \times C^{(m_n)}[a, b]$. We say that u is a solution of *fixed sign* if for each $1 \le i \le n$, we have $\theta_i u_i(t) \ge 0$ for $t \in [a, b]$ where $\theta_i \in \{1, -1\}$ is predetermined. If we choose $\theta_i = 1, 1 \le i \le n$, then our fixed-sign solution u becomes a *positive* solution, i.e., $u_i(t) \ge 0$ for $t \in [a, b], 1 \le i \le n$. Although it is only meaningful to have positive solutions in many practical problems (especially when $n = 1$), our definition of *fixed-sign* solution is more general and flexible to cater to systems whose solutions u consist of *some positive* as well as *some negative* components.

We shall establish criteria so that the system (A) has at least *three* fixed-sign solutions. Estimates on the norms of these solutions will also be provided. Knowledge of how many solutions is probably most important from a numerical standpoint. If it is known that there are multiple solutions, then naturally one may need to develop methods that produce a specific one of the solutions for efficiency sake. The main tools employed in the present work are the fixed-point theorems of Leggett and Williams [13] as well as of Avery [14]. Not only that *new* results are obtained, we also discuss the generality of the results, and illustrate their importance through some examples.

The plan of the paper is as follows. Section 2 contains the necessary definitions and fixed point theorems. The existence criteria are developed and discussed in Section 3. Finally, we present some examples in Section 4 to illustrate the importance of the results obtained.

2. Preliminaries

In this section we shall state some necessary definitions and the relevant fixed-point theorems. Let B be a Banach space equipped with norm $\|\cdot\|$.

Definition 2.1. Let C $(\subset B)$ be a nonempty closed convex set. We say that C is a *cone* provided the following conditions are satisfied:

(a) If $u \in C$ and $\alpha \ge 0$, then $\alpha u \in C$;
(b) If $u \in C$ and $-u \in C$, then $u = 0$.

Definition 2.2. Let C $(\subset B)$ be a cone. A map ψ is called a *nonnegative continuous concave functional* on C and a map β is called a *nonnegative continuous convex functional* on C if the following conditions are satisfied:

(a) $\psi, \beta : C \to [0, \infty)$ are continuous;
(b) $\psi(ty + (1-t)z) \ge t\psi(y) + (1-t)\psi(z)$ and $\beta(ty + (1-t)z) \le t\beta(y) + (1-t)\beta(z)$ for all $y, z \in C$ and $0 \le t \le 1$.

Definition 2.3. A function $P : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is a L^q-*Carathéodory function* if the following conditions hold:

(a) the map $t \to P(t, u)$ is measurable for all $u \in \mathbb{R}^n$;

(b) the map $u \to P(t,u)$ is continuous for almost all $t \in [a,b]$;
(c) for any $r > 0$, there exists $\mu_r \in L^q[a,b]$ such that $|u| \le r$ implies that $|P(t,u)| \le \mu_r(t)$ for almost all $t \in [a,b]$.

Let γ, β, Θ be nonnegative continuous convex functionals on C and α, ψ be nonnegative continuous concave functionals on C. For nonnegative numbers w_1, w_2, w_3, we shall introduce the following notations:

$$C(w_1) = \{u \in C \mid \|u\| < w_1\},$$
$$C(\psi, w_1, w_2) = \{u \in C \mid \psi(u) \ge w_1 \text{ and } \|u\| \le w_2\},$$
$$P(\gamma, w_1) = \{u \in C \mid \gamma(u) < w_1\},$$
$$P(\gamma, \alpha, w_1, w_2) = \{u \in C \mid \alpha(u) \ge w_1 \text{ and } \gamma(u) \le w_2\},$$
$$Q(\gamma, \beta, w_1, w_2) = \{u \in C \mid \beta(u) \le w_1 \text{ and } \gamma(u) \le w_2\},$$
$$P(\gamma, \Theta, \alpha, w_1, w_2, w_3) = \{u \in C \mid \alpha(u) \ge w_1,\ \Theta(u) \le w_2 \text{ and } \gamma(u) \le w_3\},$$
$$Q(\gamma, \beta, \psi, w_1, w_2, w_3) = \{u \in C \mid \psi(u) \ge w_1,\ \beta(u) \le w_2 \text{ and } \gamma(u) \le w_3\}.$$

The following fixed-point theorems are needed later. The first is the *Leggett–Williams' fixed-point theorem*, and the second is the *five-functional fixed-point theorem*.

Theorem 2.1 ([13]). *Let C ($\subset B$) be a cone, and $w_4 > 0$ be given. Assume that ψ is a nonnegative continuous concave functional on C such that $\psi(u) \le \|u\|$ for all $u \in \overline{C}(w_4)$, and let $S : \overline{C}(w_4) \to \overline{C}(w_4)$ be a continuous and completely continuous operator. Suppose that there exist numbers w_1, w_2, w_3 with $0 < w_1 < w_2 < w_3 \le w_4$ such that*

(a) *$\{u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2\} \ne \emptyset$, and $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_3)$;*
(b) *$\|Su\| < w_1$ for all $u \in \overline{C}(w_1)$;*
(c) *$\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$.*

Then, S has (at least) three fixed points u^1, u^2 and u^3 in $\overline{C}(w_4)$. Furthermore, we have

$$u^1 \in C(w_1), \quad u^2 \in \{u \in C(\psi, w_2, w_4) \mid \psi(u) > w_2\} \quad \text{and} \quad u^3 \in \overline{C}(w_4) \setminus \left(C(\psi, w_2, w_4) \cup \overline{C}(w_1)\right). \tag{2.1}$$

Theorem 2.2 ([14]). *Let C ($\subset B$) be a cone. Assume that there exist positive numbers w_5, M, nonnegative continuous convex functionals γ, β, Θ on C, and nonnegative continuous concave functionals α, ψ on C, with $\alpha(u) \le \beta(u)$ and $\|u\| \le M\gamma(u)$ for all $u \in \overline{P}(\gamma, w_5)$. Let $S : \overline{P}(\gamma, w_5) \to \overline{P}(\gamma, w_5)$ be a continuous and completely continuous operator. Suppose that there exist nonnegative numbers w_1, w_2, w_3, w_4 with $0 < w_2 < w_3$ such that:*

(a) *$\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\} \ne \emptyset$, and $\alpha(Su) > w_3$ for all $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$;*
(b) *$\{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\} \ne \emptyset$, and $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$;*
(c) *$\alpha(Su) > w_3$ for all $u \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Su) > w_4$;*
(d) *$\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Su) < w_1$.*

Then, S has (at least) three fixed points u^1, u^2 and u^3 in $\overline{P}(\gamma, w_5)$. Furthermore, we have

$$\beta(u^1) < w_2, \qquad \alpha(u^2) > w_3, \quad \text{and} \quad \beta(u^3) > w_2 \quad \text{with } \alpha(u^3) < w_3. \tag{2.2}$$

3. Main results

Throughout we shall denote $u = (u_1, u_2, \ldots, u_n)$. Let the Banach space

$$B = \{u \in C^{(m_1)}[a,b] \times C^{(m_2)}[a,b] \times \cdots \times C^{(m_n)}[a,b] \mid u_i^{(j)}(a) = 0, 0 \le j \le m_i - 4, 1 \le i \le n\}$$

be equipped with the norm

$$\|u\| = \max_{1 \le i \le n} \sup_{t \in [a,b]} |u_i^{(m_i-3)}(t)| = \max_{1 \le i \le n} |u_i|_0$$

where we denote $|u_i|_0 = \sup_{t \in [a,b]} |u_i^{(m_i-3)}(t)|,\ 1 \le i \le n$.

To apply the fixed point theorems in Section 2, we need to define an operator $S : B \to B$ so that a solution u of the system (A) is a fixed point of S, i.e. $u = Su$. Let $\overline{G}_i(t,s)$ be the Green's function for the boundary value problem

$$\begin{aligned} & y^{(m_i)}(t) = 0, \quad t \in [a,b] \\ & y^{(j)}(a) = 0, \quad 0 \le j \le m_i - 3 \\ & y^{(m_i-2)}(t^*) = 0, \qquad \xi y^{(m_i-3)}(b) + \delta y^{(m_i-1)}(b) = 0. \end{aligned} \tag{3.1}$$

We shall define the operator $S : B \to B$ by

$$\begin{aligned} & Su(t) = (S_1u(t), S_2u(t), \ldots, S_nu(t)), \quad t \in [a,b] \\ & S_iu(t) = \int_a^b \overline{G}_i(t,s) f_i(s, u_1(\phi_1(s)), u_2(\phi_2(s)), \ldots, u_n(\phi_n(s)))\mathrm{d}s = \int_a^b \overline{G}_i(t,s) f_i(s, u(\phi(s)))\mathrm{d}s, \\ & \qquad t \in [a,b],\ 1 \le i \le n \end{aligned} \tag{3.2}$$

where we denote $u(\phi(s)) = (u_1(\phi_1(s)), u_2(\phi_2(s)), \ldots, u_n(\phi_n(s)))$. Clearly, a fixed point of the operator S is a solution of the system (A).

Let

$$g(t,s) = \frac{\partial^{m_i-3}}{\partial t^{m_i-3}} \overline{G}_i(t,s). \tag{3.3}$$

It can be verified that $g(t,s)$ is the Green's function for the boundary value problem

$$y'''(t) = 0, \quad t \in [a,b]; \qquad y(a) = y'(t^*) = 0, \qquad \xi y(b) + \delta y''(b) = 0. \tag{3.4}$$

Our first lemma gives the properties of the Green's function $g(t,s)$ which will be used later.

Lemma 3.1 ([5]). *It is known that for $t, s \in [a,b]$,*

$$g(t,s) = \begin{cases} s \in [a,t^*] : & \begin{cases} \dfrac{t-a}{2}(2s-t-a) + \dfrac{\xi(t-a)}{2\eta}(s-a)^2(2t^*-a-t), & t \le s \\ \dfrac{(s-a)^2}{2\eta}[\eta + \xi(t-a)(2t^*-a-t)], & t \ge s \end{cases} \\ s \in [t^*,b] : & \begin{cases} \dfrac{t-a}{2\eta}(2t^*-a-t)[2\delta + \xi(b-s)^2], & t \le s \\ \dfrac{t-a}{2\eta}(2t^*-a-t)[2\delta + \xi(b-s)^2] + \dfrac{(t-s)^2}{2}, & t \ge s. \end{cases} \end{cases}$$

Moreover,

$$g(t,s) \ge 0, \quad t,s \in [a,b]; \qquad g(t,s) > 0, \quad t,s \in (a,b] \tag{3.5}$$

$$g(t,s) \le g(t^*,s), \quad t,s \in [a,b] \tag{3.6}$$

$$g(t,s) \ge Mg(t^*,s), \quad t \in [t^*-h, t^*+h],\ s \in [a,b] \tag{3.7}$$

where $h \in (0, b-t^*)$ is fixed and $M = \frac{(t^*-a+h)(t^*-a-h)}{(t^*-a)^2} \in (0,1)$.

We shall list the conditions that are needed later. Note that in these conditions $\theta_i \in \{1,-1\}$, $1 \le i \le n$ are fixed,

$$[0,\infty)_i = \begin{cases} [0,\infty), & \text{if } \theta_i = 1 \\ (-\infty,0], & \text{if } \theta_i = -1 \end{cases}$$

$$\tilde{K} = \{u \in B \mid \text{ for each } 1 \le i \le n, \theta_i u_i(t) \ge 0 \text{ for } t \in [a,b]\},$$

$$K = \{u \in \tilde{K} \mid \text{ for some } j \in \{1,2,\ldots,n\}, \theta_j u_j(t) > 0 \text{ for some } t \in [a,b]\} = \tilde{K} \setminus \{0\}.$$

(C1) For each $1 \le i \le n$, $f_i : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function.

(C2) For each $1 \le i \le n$,

$$\theta_i f_i(t,u) \ge 0, \quad u \in \tilde{K}, \text{ a.e. } t \in (a,b) \quad \text{and} \quad \theta_i f_i(t,u) > 0, \quad u \in K, \text{ a.e. } t \in (a,b).$$

(C3) For each $1 \le i \le n$, ϕ_i is continuous and ϕ_i maps $[a,b]$ into $[a,b]$.

(C4) There exist continuous functions $p, \nu, \mu_i, 1 \le i \le n$ with $p : \prod_{j=1}^{n}[0,\infty)_j \to [0,\infty)$ and $\nu, \mu_i : (a,b) \to [0,\infty)$ such that for each $1 \le i \le n$,

$$\mu_i(t)p(u) \le \theta_i f_i(t,u) \le \nu(t)p(u), \quad u \in \tilde{K}, \text{ a.e. } t \in (a,b).$$

(C5) For each $1 \le i \le n$, there exists a number $0 < \rho_i \le 1$ such that

$$\mu_i(t) \ge \rho_i \nu(t), \quad \text{a.e. } t \in (a,b).$$

Remark 3.1. There are many examples of a deviating function ϕ_i satisfying (C3). For instance, when $a = 0$ and $b = 1$, $\phi_i(t) = t^2, \ \sqrt{1-t}, \ \cos\frac{\pi t}{2}$.

Lemma 3.2. *Let* (C1) *hold. Then, the operator S defined in (3.2) is continuous and completely continuous.*

Proof. From Lemma 3.1, we have $g(t,s) \equiv g^t(s) \in C[a,b] \subseteq L^\infty[a,b], \ t \in [a,b]$ and the map $t \to g(t,s)$ is continuous from $[a,b]$ to $C[a,b]$. This together with $f_i : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function ensures that S is continuous and completely continuous. $\square$

Let $h \in (0, b - t^*)$ be fixed. We define a cone C in B as

$$C = \left\{ u \in B \mid \text{ for each } 1 \le i \le n, \theta_i u_i^{(m_i-3)}(t) \ge 0 \text{ for } t \in [a,b], \text{ and } \min_{t\in[t^*-h,t^*+h]} \theta_i u_i^{(m_i-3)}(t) \ge M|u_i|_0 \right\}$$

where M is defined in (3.7). In the subsequent development, we will always assume that $h \in (0, b - t^*)$ is fixed.

Our next lemma presents some properties of B and C that will be used later.

Lemma 3.3. (a) *Let $u \in B$. Then, for each $1 \le i \le n$ we have*

$$|u_i^{(j)}(t)| \le \frac{(t-a)^{m_i-3-j}}{(m_i-3-j)!}\,|u_i|_0, \quad t \in [a,b], \ 0 \le j \le m_i - 3. \tag{3.8}$$

In particular,

$$|u_i(t)| \le \frac{(b-a)^{m_i-3}}{(m_i-3)!}\,\|u\|, \quad t \in [a,b], \ 1 \le i \le n. \tag{3.9}$$

(b) *Let $u \in C$. Then, for each $1 \le i \le n$ we have*

$$\theta_i u_i^{(j)}(t) \ge 0, \quad t \in [a,b], \ 0 \le j \le m_i - 3 \tag{3.10}$$

and

$$\theta_i u_i^{(j)}(t) \ge \frac{[t-(t^*-h)]^{m_i-3-j}}{(m_i-3-j)!} M|u_i|_0, \quad t \in [t^*-h, t^*+h], \ 0 \le j \le m_i - 3. \tag{3.11}$$

In particular,

$$\theta_i u_i(t) \ge \frac{h^{m_i-3}}{(m_i-3)!}\, M|u_i|_0, \quad t \in [t^*, t^*+h]. \tag{3.12}$$

Proof. (a) Let $u \in B$. Then, for each $1 \le i \le n$ we have

$$u_i^{(m_i-4)}(t) = \int_a^t u_i^{(m_i-3)}(s)\mathrm{d}s, \quad t \in [a,b]$$

which implies

$$|u_i^{(m_i-4)}(t)| \le (t-a)|u_i|_0, \quad t \in [a,b].$$

Using the above inequality in

$$u_i^{(m_i-5)}(t) = \int_a^t u_i^{(m_i-4)}(s)\mathrm{d}s, \quad t \in [a,b]$$

yields

$$|u_i^{(m_i-5)}(t)| \le \int_a^t (s-a)|u_i|_0 \, \mathrm{d}s = \frac{(t-a)^2}{2!}|u_i|_0, \quad t \in [a,b].$$

Continuing in the same manner we obtain (3.8).

(b) Let $u \in C$. Inequality (3.10) is obvious from the fact that $\theta_i u_i^{(m_i-3)}(t) \ge 0$ for $t \in [a,b]$ and the relation

$$u_i^{(j)}(t) = \int_a^t u_i^{(j+1)}(s)\mathrm{d}s, \quad t \in [a,b], 0 \le j \le m_i - 4.$$

Next, to prove (3.11), we have for $1 \le i \le n$ and $t \in [t^* - h, t^* + h]$,

$$\theta_i u_i^{(m_i-4)}(t) = \int_a^t \theta_i u_i^{(m_i-3)}(s)\mathrm{d}s \ge \int_{t^*-h}^t \theta_i u_i^{(m_i-3)}(s)\mathrm{d}s \ge \int_{t^*-h}^t M|u_i|_0 \, \mathrm{d}s = [t-(t^*-h)]M|u_i|_0.$$

It follows that for $1 \le i \le n$ and $t \in [t^* - h, t^* + h]$,

$$\begin{aligned}\theta_i u_i^{(m_i-5)}(t) &= \int_a^t \theta_i u_i^{(m_i-4)}(s)\mathrm{d}s \ge \int_{t^*-h}^t \theta_i u_i^{(m_i-4)}(s)\mathrm{d}s \\ &\ge \int_{t^*-h}^t [s-(t^*-h)]M|u_i|_0\mathrm{d}s = \frac{[t-(t^*-h)]^2}{2!}M|u_i|_0.\end{aligned}$$

Continuing the process we obtain (3.11). Inequality (3.12) is immediate from (3.11) by taking $j = 0$ and substituting $t = t^*$ in the right side of (3.11). $\square$

Remark 3.2. The inequality (3.10) implies that $C \subset \tilde{K}$. Hence, a solution obtained in the cone C will be a *fixed-sign* solution.

Lemma 3.4. *Let* (C1)–(C3) *hold. Then, the operator S maps C into C.*

Proof. Let $u \in C$. Using (3.2), (3.3) and (3.5), (C2), (C3) and the fact that $C \subset \tilde{K}$, we have

$$\theta_i(S_i u)^{(m_i-3)}(t) = \int_a^b g(t,s)\theta_i f_i(s, u(\phi(s)))\mathrm{d}s \ge 0, \quad t \in [a,b], \ 1 \le i \le n. \tag{3.13}$$

Applying (3.6), it follows that

$$|S_i u|_0 = \sup_{t\in[a,b]} \theta_i(S_i u)^{(m_i-3)}(t) \le \int_a^b g(t^*,s)\theta_i f_i(s,u(\phi(s)))\mathrm{d}s, \quad 1 \le i \le n. \tag{3.14}$$

Now, in view of (3.13), (3.7) and (3.14), we get for $1 \le i \le n$ and $t \in [t^* - h, t^* + h]$,

$$\theta_i(S_i u)^{(m_i-3)}(t) \ge \int_a^b M g(t^*,s)\theta_i f_i(s,u(\phi(s)))\mathrm{d}s \ge M|S_i u|_0.$$

Hence,

$$\min_{t\in[t^*-h,t^*+h]} \theta_i(S_i u)^{(m_i-3)}(t) \ge M|S_i u|_0, \quad 1 \le i \le n.$$

We have shown that $Su \in C$. $\square$

Remark 3.3. If, in addition to (C1)–(C3), condition (C4) is also satisfied, then we have for $u \in C$,

$$\int_a^b g(t,s)\mu_i(s)p(u(\phi(s)))\mathrm{d}s \le \theta_i(S_i u)^{(m_i-3)}(t) \le \int_a^b g(t,s)\nu(s)p(u(\phi(s)))\mathrm{d}s, \quad t \in [a,b], \ 1 \le i \le n. \tag{3.15}$$

In view of (3.6), we get

$$|S_i u|_0 \le \int_a^b g(t^*, s)\nu(s)p(u(\phi(s)))\mathrm{d}s, \ 1 \le i \le n \tag{3.16}$$

and so

$$\|Su\| = \max_{1\le i\le n} |S_i u|_0 \le \int_a^b g(t^*, s)\nu(s)p(u(\phi(s)))\mathrm{d}s. \tag{3.17}$$

If condition (C5) is also satisfied, then by using (3.15), (3.7) and (3.16), we get for $1 \le i \le n$ and $t \in [t^* - h, t^* + h]$,

$$\theta_i (S_i u)^{(m_i-3)}(t) \ge \int_a^b Mg(t^*, s)\mu_i(s)p(u(\phi(s)))\mathrm{d}s \ge \int_a^b Mg(t^*, s)\rho_i\nu(s)p(u(\phi(s)))\mathrm{d}s \ge M\rho_i |S_i u|_0.$$

It follows that

$$\min_{t\in[t^*-h,t^*+h]} \theta_i (S_i u)^{(m_i-3)}(t) \ge M\rho_i |S_i u|_0, \quad 1 \le i \le n.$$

Hence, we see that if (C1)–(C5) are satisfied, the cone C can be replaced with another cone $C'(C \subset C')$ given by

$$C' = \left\{ u \in B \mid \text{ for each } 1 \le i \le n, \theta_i u_i^{(m_i-3)}(t) \ge 0 \text{ for } t \in [a, b], \right.$$
$$\left. \text{and } \min_{t\in[t^*-h,t^*+h]} \theta_i u_i^{(m_i-3)}(t) \ge M\rho_i |u_i|_0 \right\}.$$

If we apply (3.17) instead of (3.16), then a similar argument will lead to another possible cone $C''(\subset C')$ where

$$C'' = \left\{ u \in B \mid \text{ for each } 1 \le i \le n, \theta_i u_i^{(m_i-3)}(t) \ge 0 \text{ for } t \in [a, b], \right.$$
$$\left. \text{and } \min_{t\in[t^*-h,t^*+h]} \theta_i u_i^{(m_i-3)}(t) \ge M\rho_i \|u\| \right\}.$$

For subsequent development, we require the following definitions. For some fixed numbers $\tau_1, \tau_2, \tau_3, \tau_4 \in [a, b]$, the subsets D, E, F, G of $[a, b]$ are defined as follows:

(a) D is the largest subset of $[a, b]$ of positive measure such that $\phi_i(t) \in [t^*, t^* + h]$, $1 \le i \le n$ for all $t \in D$;
(b) E is the largest subset of $[a, b]$ of positive measure such that $\phi_i(t) \in [\tau_1, \tau_4]$, $1 \le i \le n$ for all $t \in E$;
(c) F is the largest subset of $[a, b]$ of positive measure such that $\phi_i(t) \in [\max\{\tau_2, t^*\}, \tau_3]$, $1 \le i \le n$ for all $t \in F$;
(d) G is the largest subset of $[a, b]$ of positive measure such that $\phi_i(t) \in [\max\{\tau_1, t^*\}, \tau_4]$, $1 \le i \le n$ for all $t \in G$.

We also define the following constants for $1 \le i \le n$:

$$q = \int_a^b g(t^*, s)\nu(s)\mathrm{d}s,$$
$$r_i = \min_{t\in[t^*-h,t^*+h]} \int_D g(t, s)\mu_i(s)\mathrm{d}s,$$
$$d_{1,i} = \min_{t\in[\tau_2,\tau_3]} \int_D g(t, s)\mu_i(s)\mathrm{d}s,$$
$$d_2 = \int_E g(t^*, s)\nu(s)\mathrm{d}s,$$
$$d_3 = \int_{[a,b]\setminus E} g(t^*, s)\nu(s)\mathrm{d}s = q - d_2,$$
$$d_{4,i} = \min_{t\in[\tau_2,\tau_3]} \int_F g(t, s)\mu_i(s)\mathrm{d}s,$$
$$d_5 = \int_G g(t^*, s)\nu(s)\mathrm{d}s,$$

$$d_6 = \int_{[a,b]\setminus G} g(t^*, s)\nu(s)\mathrm{d}s = q - d_5.$$

Lemma 3.5. *Let* (C1)–(C4) *hold, and assume* (C6) *the function* $g(t^*, s)\nu(s)$ *is nonzero on a subset of* $[a, b]$ *of positive measure.*

Suppose that there exists a number $d > 0$ *such that for* $|u_j| \in \left[0, \frac{(b-a)^{m_j-3}}{(m_j-3)!}\, d\right],\ 1 \le j \le n,$

$$p(u_1, u_2, \ldots, u_n) < \frac{d}{q}. \tag{3.18}$$

Then,

$$S(\overline{C}(d)) \subseteq C(d) \subset \overline{C}(d). \tag{3.19}$$

Proof. Let $u \in \overline{C}(d)$. Then, $\|u\| \le d$. It follows from (3.9) and (C3) that $|u_j(\phi_j(s))| \in \left[0, \frac{(b-a)^{m_j-3}}{(m_j-3)!}\, d\right],\ 1 \le j \le n$ for all $s \in [a, b]$. Applying (3.16), (C6) and (3.18), we find for $1 \le i \le n$,

$$|S_i u|_0 \le \int_a^b g(t^*, s)\nu(s)p(u(\phi(s)))\mathrm{d}s < \int_a^b g(t^*, s)\nu(s)\frac{d}{q}\mathrm{d}s = q\frac{d}{q} = d.$$

Thus, $\|Su\| < d$. Coupling with the fact that $Su \in C$ (Lemma 3.4), we have $Su \in C(d)$. The conclusion (3.19) is now immediate. □

The next lemma is similar to Lemma 3.5, hence we shall omit the proof.

Lemma 3.6. *Let* (C1)–(C4) *hold. Suppose that there exists a number* $d > 0$ *such that for* $|u_j| \in \left[0, \frac{(b-a)^{m_j-3}}{(m_j-3)!} d\right],\ 1 \le j \le n,$

$$p(u_1, u_2, \ldots, u_n) \le \frac{d}{q}.$$

Then,

$$S(\overline{C}(d)) \subseteq \overline{C}(d).$$

We are now ready to establish existence criteria for three fixed-sign solutions of (A). Our first result employs Leggett–Williams' fixed point theorem (Theorem 2.1).

Theorem 3.1. *Let* (C1)–(C6) *hold. Assume the set D exists, and*

(C7) *for each* $1 \le i \le n$ *and each* $t \in [t^* - h, t^* + h]$, *the function* $g(t, s)\mu_i(s) \equiv g^t(s)\mu_i(s)$ *is nonzero on a subset of* D *of positive measure.*

Suppose that there exist numbers w_1, w_2, w_3 with

$$0 < w_1 < w_2 < \frac{w_2}{M \min\limits_{1 \le i \le n} \rho_i} \le w_3$$

such that the following hold:

(P) $p(u_1, u_2, \ldots, u_n) < \frac{w_1}{q}$ for $|u_j| \in \left[0, \frac{(b-a)^{m_j-3}}{(m_j-3)!} w_1\right], 1 \le j \le n;$

(Q) one of the following holds:

(Q1) $\limsup_{|u_1|, |u_2|, \ldots, |u_n| \to \infty} \frac{p(u_1, u_2, \ldots, u_n)}{|u_k|} < \frac{1}{q}$ for some $k \in \{1, 2, \ldots, n\}$;

(Q2) there exists a number $d(\geq w_3)$ such that $p(u_1, u_2, \ldots, u_n) \leq \frac{d}{q}$ for $|u_j| \in \left[0, \frac{(b-a)^{m_j-3}}{(m_j-3)!}d\right]$, $1 \leq j \leq n$;

(R) for each $1 \leq i \leq n$, $p(u_1, u_2, \ldots, u_n) > \frac{w_2}{r_i}$ for $|u_j| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!}w_2, \frac{(b-a)^{m_j-3}}{(m_j-3)!}w_3\right]$, $1 \leq j \leq n$.

Then, the system (A) has (at least) three fixed-sign solutions $u^1, u^2, u^3 \in C$ such that

$$\begin{aligned}&\|u^1\| < w_1; \qquad \theta_i(u_i^2)^{(m_i-3)}(t) > w_2, \quad t \in \left[t^* - h, t^* + h\right], \ 1 \leq i \leq n;\\ &\|u^3\| > w_1 \quad \text{and} \quad \min_{1\leq i\leq n}\ \min_{t\in[t^*-h,t^*+h]} \theta_i(u_i^3)^{(m_i-3)}(t) < w_2.\end{aligned} \tag{3.20}$$

Proof. We shall employ Theorem 2.1. The operator S is continuous and completely continuous by Lemma 3.2. First, we shall prove that condition (Q) implies the existence of a number w_4, where $w_4 \geq w_3$, such that

$$S(\overline{C}(w_4)) \subseteq \overline{C}(w_4). \tag{3.21}$$

If (Q2) holds, then by Lemma 3.6 we immediately have (3.21) where we pick $w_4 = d$. On the other hand, if (Q1) is satisfied, then there exist $N > 0$ and $\epsilon < \frac{1}{q}$ such that

$$\frac{p(u_1, u_2, \ldots, u_n)}{|u_k|} < \epsilon, \qquad |u_1|, |u_2|, \ldots, |u_n| > N \tag{3.22}$$

where k is as in (Q1). Define

$$L = \max_{|u_j|\in[0,N],1\leq j\leq n} p(u_1, u_2, \ldots, u_n).$$

Together with (3.22), we have for all $(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$,

$$p(u_1, u_2, \ldots, u_n) \leq L + \epsilon|u_k| \tag{3.23}$$

where k is as in (Q1). Now, pick the number w_4 so that

$$w_4 > \max\left\{w_3, L\left[\frac{1}{q} - \epsilon \max_{1\leq j\leq n} \frac{(b-a)^{m_j-3}}{(m_j-3)!}\right]^{-1}\right\}. \tag{3.24}$$

Let $u \in \overline{C}(w_4)$. Using (3.16), (3.23), (C3), (3.9) and (3.24), we get for $1 \leq i \leq n$,

$$\begin{aligned}|S_i u|_0 &\leq \int_a^b g(t^*, s)\nu(s)p(u(\phi(s)))\mathrm{d}s\\ &\leq \int_a^b g(t^*, s)\nu(s)(L + \epsilon|u_k(\phi_k(s))|)\mathrm{d}s\\ &\leq \int_a^b g(t^*, s)\nu(s)\left[L + \epsilon\frac{(b-a)^{m_k-3}}{(m_k-3)!}\|u\|\right]\mathrm{d}s\\ &\leq q\left[L + \epsilon \max_{1\leq j\leq n} \frac{(b-a)^{m_j-3}}{(m_j-3)!}w_4\right]\\ &< q\left\{w_4\left[\frac{1}{q} - \epsilon \max_{1\leq j\leq n} \frac{(b-a)^{m_j-3}}{(m_j-3)!}\right] + \epsilon \max_{1\leq j\leq n} \frac{(b-a)^{m_j-3}}{(m_j-3)!}w_4\right\} = w_4.\end{aligned}$$

Hence, $\|Su\| < w_4$ and so $Su \in C(w_4) \subset \overline{C}(w_4)$. Thus, (3.21) follows immediately.

Let $\psi : C \to [0, \infty)$ be defined by

$$\psi(u) = \min_{1\leq i\leq n}\ \min_{t\in[t^*-h,t^*+h]} \theta_i u_i^{(m_i-3)}(t).$$

Clearly, ψ is a nonnegative continuous concave functional on C and $\psi(u) \leq \|u\|$ for all $u \in C$.

We shall verify that condition (a) of Theorem 2.1 is satisfied. The set $\{u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2\} \neq \emptyset$ since it contains the element

$$\left(\frac{\theta_1 t^{m_1-3}}{(m_1-3)!}\frac{w_2+w_3}{2}, \frac{\theta_2 t^{m_2-3}}{(m_2-3)!}\frac{w_2+w_3}{2}, \ldots, \frac{\theta_n t^{m_n-3}}{(m_n-3)!}\frac{w_2+w_3}{2}\right).$$

Let $u \in C(\psi, w_2, w_3)$. Then, $w_2 \le \psi(u) \le \|u\| \le w_3$. Note that $\phi_j(s) \in [t^*, t^*+h],\ 1 \le j \le n$ for all $s \in D$, and also $|u_j|_0 \ge \psi(u) \ge w_2,\ 1 \le j \le n$. Thus, it follows from (3.9) and (3.12) that

$$\theta_j u_j(\phi_j(s)) = |u_j(\phi_j(s))| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!}\, w_2, \frac{(b-a)^{m_j-3}}{(m_j-3)!}\, w_3\right], \quad s \in D,\ 1 \le j \le n. \tag{3.25}$$

Applying (3.15), (C7), (3.25) and (R), we get

$$\begin{aligned}
\psi(Su) &= \min_{1\le i\le n}\min_{t\in[t^*-h,t^*+h]} \theta_i (S_i u)^{(m_i-3)}(t)\\
&\ge \min_{1\le i\le n}\min_{t\in[t^*-h,t^*+h]} \int_a^b g(t,s)\mu_i(s)p(u(\phi(s)))\mathrm{d}s\\
&\ge \min_{1\le i\le n}\min_{t\in[t^*-h,t^*+h]} \int_D g(t,s)\mu_i(s)p(u(\phi(s)))\mathrm{d}s\\
&> \min_{1\le i\le n}\min_{t\in[t^*-h,t^*+h]} \int_D g(t,s)\mu_i(s)\frac{w_2}{r_i}\mathrm{d}s = \min_{1\le i\le n}\frac{r_i}{r_i}w_2 = w_2.
\end{aligned}$$

Therefore, we have shown that $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_3)$.

Next, by Lemma 3.5 and condition (P), we have $S(\overline{C}(w_1)) \subseteq C(w_1)$. Hence, condition (b) of Theorem 2.1 is fulfilled.

It remains to show that condition (c) of Theorem 2.1 is satisfied. Let $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$. Using (3.15), (3.7), (C5) and (3.17), we find

$$\begin{aligned}
\psi(Su) &\ge \min_{1\le i\le n}\min_{t\in[t^*-h,t^*+h]} \int_a^b g(t,s)\mu_i(s)p(u(\phi(s)))\mathrm{d}s\\
&\ge \min_{1\le i\le n}\int_a^b Mg(t^*,s)\rho_i\nu(s)p(u(\phi(s)))\mathrm{d}s \ge \min_{1\le i\le n} M\rho_i\|Su\| > \min_{1\le i\le n} M\rho_i w_3 \ge w_2.
\end{aligned}$$

Hence, we have proved that $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$.

It now follows from Theorem 2.1 that the system (A) has (at least) three *fixed-sign* solutions $u^1, u^2, u^3 \in \overline{C}(w_4)$ satisfying (2.1), which readily reduces to (3.20). □

We shall now employ the five-functional fixed-point theorem (Theorem 2.2) to give other existence criteria. In applying Theorem 2.2 it is possible to choose the functionals and constants in many different ways. We shall present two results to show the arguments involved. In particular the first result is a generalization of Theorem 3.1.

Theorem 3.2. *Let* (C1)–(C5) *hold. Let the numbers* $\tau_1, \tau_2, \tau_3, \tau_4$ *satisfying*

$$a \le \tau_1 \le t^* - h \le \tau_2 < \tau_3 \le t^* + h \le \tau_4 \le b$$

be such that the sets D and E exist. Assume:

(C8) *for each $1 \le i \le n$ and each $t \in [\tau_2, \tau_3]$, the function $g(t,s)\mu_i(s) \equiv g^t(s)\mu_i(s)$ is nonzero on a subset of D of positive measure;*

(C9) *the function $g(t^*, s)\nu(s)$ is nonzero on a subset of E of positive measure.*

Suppose that there exist numbers w_2, w_3, w_4, w_5 with

$$0 < w_2 < w_3 < \frac{w_3}{M\min_{1\le i\le n}\rho_i} \le w_4 \le w_5 \quad \text{and} \quad w_2 > \frac{w_5 d_3}{q}$$

such that the following hold:

(P′) $p(u_1, u_2, \ldots, u_n) < \frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right)$ *for* $|u_j| \in [0, \ell_j(w_2)],\ 1 \le j \le n$ *where*

$$\ell_j(x) = \sum_{k=0}^{m_j-4} \frac{(\tau_1 - a)^{m_j-3-k}}{(m_j-3-k)!} \frac{(\tau_4-\tau_1)^k}{k!} w_5 + \frac{(\tau_4-\tau_1)^{m_j-3}}{(m_j-3)!} x;$$

(Q′) $p(u_1, u_2, \ldots, u_n) \le \frac{w_5}{q}$ *for* $|u_j| \in \left[0, \frac{(b-a)^{m_j-3}}{(m_j-3)!} w_5\right],\ 1 \le j \le n;$

(R′) *for each* $1 \le i \le n$, $p(u_1, u_2, \ldots, u_n) > \frac{w_3}{d_{1,i}}$ *for* $|u_j| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!} w_3, \ell_j(w_4)\right],\ 1 \le j \le n.$

Then, the system (A) *has (at least) three fixed-sign solutions* $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$\begin{aligned} &\theta_i (u_i^1)^{(m_i-3)}(t) < w_2, \quad t \in [\tau_1, \tau_4],\ 1 \le i \le n; \qquad \theta_i (u_i^2)^{(m_i-3)}(t) > w_3, \quad t \in [\tau_2, \tau_3],\ 1 \le i \le n; \\ &\max_{1\le i\le n} \max_{t\in[\tau_1,\tau_4]} \theta_i (u_i^3)^{(m_i-3)}(t) > w_2 \quad \text{and} \quad \min_{1\le i\le n} \min_{t\in[\tau_2,\tau_3]} \theta_i (u_i^3)^{(m_i-3)}(t) < w_3. \end{aligned} \tag{3.26}$$

Proof. To apply Theorem 2.2, we shall define the following functionals on C :

$$\begin{aligned} &\gamma(u) = \|u\|, \qquad \beta(u) = \Theta(u) = \max_{1\le i\le n} \max_{t\in[\tau_1,\tau_4]} \theta_i u_i^{(m_i-3)}(t), \\ &\psi(u) = \min_{1\le i\le n} \min_{t\in[t^*-h, t^*+h]} \theta_i u_i^{(m_i-3)}(t), \qquad \alpha(u) = \min_{1\le i\le n} \min_{t\in[\tau_2,\tau_3]} \theta_i u_i^{(m_i-3)}(t). \end{aligned} \tag{3.27}$$

The operator S is continuous and completely continuous by Lemma 3.2. First, we shall show that the operator S maps $\overline{P}(\gamma, w_5)$ into $\overline{P}(\gamma, w_5)$. Let $u \in \overline{P}(\gamma, w_5)$. Then, $\|u\| \le w_5$ and in view of (3.9), we have $|u_j| \in \left[0, \frac{(b-a)^{m_j-3}}{(m_j-3)!} w_5\right],\ 1 \le j \le n$. Using (Q′) and Lemma 3.6 immediately gives $S(\overline{P}(\gamma, w_5)) \subseteq \overline{P}(\gamma, w_5)$.

Next, to see that condition (a) of Theorem 2.2 is fulfilled, we note that $\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\} \ne \emptyset$ since it contains the element

$$\left(\frac{\theta_1 t^{m_1-3}}{(m_1-3)!} \frac{w_3+w_4}{2}, \frac{\theta_2 t^{m_2-3}}{(m_2-3)!} \frac{w_3+w_4}{2}, \ldots, \frac{\theta_n t^{m_n-3}}{(m_n-3)!} \frac{w_3+w_4}{2}\right).$$

Let $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$. Then, we have $\alpha(u) \ge w_3$ which implies $|u_j|_0 \ge w_3$, $1 \le j \le n$. In view of (3.12), it follows that

$$\theta_j u_j(s) \ge \frac{Mh^{m_j-3}}{(m_j-3)!} w_3, \quad s \in [t^*, t^*+h],\ 1 \le j \le n. \tag{3.28}$$

Also, we have $\Theta(u) \le w_4$ which implies

$$\theta_j u_j^{(m_j-3)}(s) \le w_4, \quad s \in [\tau_1, \tau_4],\ 1 \le j \le n.$$

Integrating the above from τ_1 to s repeatedly yields

$$\theta_j u_j(s) \le \sum_{k=0}^{m_j-4} \theta_j u_j^{(k)}(\tau_1) \frac{(s-\tau_1)^k}{k!} + \frac{(s-\tau_1)^{m_j-3}}{(m_j-3)!} w_4, \quad s \in [\tau_1, \tau_4],\ 1 \le j \le n$$

which, on applying (3.8), leads to

$$\theta_j u_j(s) \le \sum_{k=0}^{m_j-4} \frac{(\tau_1-a)^{m_j-3-k}}{(m_j-3-k)!} \frac{(\tau_4-\tau_1)^k}{k!} w_5 + \frac{(\tau_4-\tau_1)^{m_j-3}}{(m_j-3)!} w_4 = \ell_j(w_4), \quad s \in [\tau_1, \tau_4],\ 1 \le j \le n. \tag{3.29}$$

Coupling (3.28) and (3.29), we get

$$|u_j(s)| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!} w_3, \ell_j(w_4)\right], \quad s \in [t^*, t^*+h],\ 1 \le j \le n$$

or equivalently

$$|u_j(\phi_j(s))| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!}w_3,\ \ell_j(w_4)\right], \quad s \in D,\ 1 \le j \le n. \tag{3.30}$$

Applying (3.15), (C8), (3.30) and (R′), we obtain

$$\begin{aligned}
\alpha(Su) &= \min_{1\le i\le n}\min_{t\in[\tau_2,\tau_3]} \theta_i (S_i u)^{(m_i-3)}(t) \\
&\ge \min_{1\le i\le n}\min_{t\in[\tau_2,\tau_3]} \int_a^b g(t,s)\mu_i(s)p(u(\phi(s)))\mathrm{d}s \\
&\ge \min_{1\le i\le n}\min_{t\in[\tau_2,\tau_3]} \int_D g(t,s)\mu_i(s)p(u(\phi(s)))\mathrm{d}s \\
&> \min_{1\le i\le n}\min_{t\in[\tau_2,\tau_3]} \int_D g(t,s)\mu_i(s)\frac{w_3}{d_{1,i}}\mathrm{d}s = \min_{1\le i\le n}\frac{d_{1,i}}{d_{1,i}}w_3 = w_3.
\end{aligned}$$

Hence, $\alpha(Su) > w_3$ for all $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$.

We shall now verify that condition (b) of Theorem 2.2 is satisfied. Let w_1 be such that $0 < w_1 < w_2$. Note that $\{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\} \ne \emptyset$ since it contains the element

$$\left(\frac{\theta_1 t^{m_1-3}}{(m_1-3)!}\frac{w_1+w_2}{2}, \frac{\theta_2 t^{m_2-3}}{(m_2-3)!}\frac{w_1+w_2}{2}, \ldots, \frac{\theta_n t^{m_n-3}}{(m_n-3)!}\frac{w_1+w_2}{2}\right).$$

Let $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$. Then, we have $\beta(u) \le w_2$ which implies

$$\theta_j u_j^{(m_j-3)}(s) \le w_2, \quad s \in [\tau_1, \tau_4],\ 1 \le j \le n.$$

As before, we integrate the above from τ_1 to s repeatedly and then apply (3.8) to get

$$\theta_j u_j(s) \le \sum_{k=0}^{m_j-4} \frac{(\tau_1-a)^{m_j-3-k}}{(m_j-3-k)!}\frac{(\tau_4-\tau_1)^k}{k!}w_5 + \frac{(\tau_4-\tau_1)^{m_j-3}}{(m_j-3)!}w_2 = \ell_j(w_2), \quad s \in [\tau_1,\tau_4],\ 1 \le j \le n \tag{3.31}$$

or equivalently

$$|u_j(\phi_j(s))| \le \ell_j(w_2), \quad s \in E,\ 1 \le j \le n. \tag{3.32}$$

Further, noting (C3), (3.9) and $\gamma(u) = \|u\| \le w_5$, we have

$$|u_j(\phi_j(s))| \le \frac{(b-a)^{m_j-3}}{(m_j-3)!}w_5, \quad s \in [a,b],\ 1 \le j \le n. \tag{3.33}$$

Applying (3.15), (3.6), (C9), (3.32), (3.33), (P′) and (Q′), we find

$$\begin{aligned}
\beta(Su) &= \max_{1\le i\le n}\max_{t\in[\tau_1,\tau_4]} \theta_i (S_i u)^{(m_i-3)}(t) \\
&\le \int_a^b g(t^*,s)\nu(s)p(u(\phi(s)))\mathrm{d}s \\
&= \int_E g(t^*,s)\nu(s)p(u(\phi(s)))\mathrm{d}s + \int_{[a,b]\setminus E} g(t^*,s)\nu(s)p(u(\phi(s)))\mathrm{d}s \\
&< \int_E g(t^*,s)\nu(s)\frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right)\mathrm{d}s + \int_{[a,b]\setminus E} g(t^*,s)\nu(s)\frac{w_5}{q}\mathrm{d}s \\
&= d_2\frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right) + d_3\frac{w_5}{q} = w_2.
\end{aligned}$$

Therefore, $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$.

Next, to show that condition (c) of Theorem 2.2 is fulfilled, we observe, by (3.15) and (3.6), that for $u \in C$,

$$\Theta(Su) = \max_{1\le i\le n} \max_{t\in[\tau_1,\tau_4]} \theta_i (S_i u)^{(m_i-3)}(t) \le \int_a^b g(t^*, s)\nu(s)p(u(\phi(s)))\mathrm{d}s. \tag{3.34}$$

Moreover, using (3.15), (3.7) and (C5), we get for $u \in C$,

$$\alpha(Su) \ge \min_{1\le i\le n} \min_{t\in[\tau_2,\tau_3]} \int_a^b g(t, s)\mu_i(s)p(u(\phi(s)))\mathrm{d}s \ge \min_{1\le i\le n} \int_a^b Mg(t^*, s)\rho_i\nu(s)p(u(\phi(s)))\mathrm{d}s. \tag{3.35}$$

Combining (3.34) and (3.35) yields

$$\alpha(Su) \ge \min_{1\le i\le n} M\rho_i\,\Theta(Su), \quad u \in C. \tag{3.36}$$

Let $u \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Su) > w_4$. Then, it follows from (3.36) that

$$\alpha(Su) \ge \min_{1\le i\le n} M\rho_i\ \Theta(Su) > \min_{1\le i\le n} M\rho_i w_4 \ge \min_{1\le i\le n} M\rho_i \frac{w_3}{\min\limits_{1\le i\le n} M\rho_i} = w_3.$$

Thus, $\alpha(Su) > w_3$ for all $u \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Su) > w_4$.

Finally, we shall prove that condition (d) of Theorem 2.2 is met. Let $u \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Su) < w_1$. Then, we have $\beta(u) \le w_2$ and $\gamma(u) \le w_5$ which give (3.32) and (3.33) respectively. Proceeding as in proving condition (b), we get $\beta(Su) < w_2$. Hence, condition (d) of Theorem 2.2 is satisfied.

It now follows from Theorem 2.2 that the system (A) has (at least) three *fixed-sign* solutions $u^1, u^2, u^3 \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$ satisfying (2.2), which reduces to (3.27) immediately. □

Remark 3.4. Consider the special case when

$$\tau_1 = a, \qquad \tau_2 = t^* - h, \qquad \tau_3 = t^* + h \quad \text{and} \quad \tau_4 = b. \tag{3.37}$$

Then, the set $E = [a, b]$ (because of (C3)), and we have

$$d_{1,i} = r_i, \quad 1 \le i \le n, \qquad d_2 = q, \qquad d_3 = 0, \qquad \text{(C8)} = \text{(C7)} \quad \text{and} \quad \text{(C9)} = \text{(C6)}. \tag{3.38}$$

It is clear that Theorem 3.2 reduces to Theorem 3.1. Hence, Theorem 3.2 is more general than Theorem 3.1. This also shows that the five-functional fixed-point theorem (Theorem 2.2), which is used to obtain Theorem 3.2, generalizes Leggett–Williams' fixed-point theorem (Theorem 2.1), which is the main tool for Theorem 3.1.

Leggett–Williams' fixed-point theorem is well known in the literature, possibly because of the ease to apply and also it produces easily verifiable criteria, as evidenced by the proof and the conditions of Theorem 3.1. In fact till today many authors, e.g. [11,15–18] are still finding new applications of this theorem. On the other hand, greater skill is needed to apply five-functional fixed-point theorem and the criteria obtained are more difficult to check, as seen from the proof and the conditions of Theorem 3.2. Nevertheless, a number of works, e.g. [18–20] have made good use of this theorem.

The next result illustrates another application of Theorem 2.2.

Theorem 3.3. *Let* (C1)–(C5) *hold. Let the numbers* $\tau_1, \tau_2, \tau_3, \tau_4$ *satisfying*

$$t^* - h \le \tau_1 \le \tau_2 < \tau_3 \le \tau_4 \le t^* + h \quad \text{and} \quad \tau_3 > \max\{\tau_2, t^*\}$$

be such that the sets F *and* G *exist. Assume*

(C10) *for each* $1 \le i \le n$ *and each* $t \in [\tau_2, \tau_3]$, *the function* $g(t, s)\mu_i(s) \equiv g^t(s)\mu_i(s)$ *is nonzero on a subset of* F *of positive measure;*

(C11) *the function* $g(t^*, s)\nu(s)$ *is nonzero on a subset of* G *of positive measure.*

Suppose that there exist numbers w_1, w_2, w_3, w_4, w_5 with

$$0 < w_1 \le w_2 \cdot M \min_{1\le i\le n} \rho_i < w_2 < w_3 < \frac{w_3}{M \min\limits_{1\le i\le n} \rho_i} \le w_4 \le w_5 \quad \text{and} \quad w_2 > \frac{w_5 d_6}{q}$$

such that the following hold:

(P″) $p(u_1, u_2, \ldots, u_n) < \frac{1}{d_5}\left(w_2 - \frac{w_5 d_6}{q}\right)$ for $|u_j| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!}\, w_1,\ \ell_j(w_2)\right],\ 1 \le j \le n$ where $\ell_j(x)$ is defined in Theorem 3.2;

(Q″) $p(u_1, u_2, \ldots, u_n) \le \frac{w_5}{q}$ for $|u_j| \in \left[0,\ \frac{(b-a)^{m_j-3}}{(m_j-3)!}\, w_5\right],\ 1 \le j \le n$;

(R″) for each $1 \le i \le n$, $p(u_1, u_2, \ldots, u_n) > \frac{w_3}{d_{4,i}}$ for $|u_j| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!} w_3, \overline{\ell}_j(w_4)\right],\ 1 \le j \le n$ where

$$\overline{\ell}_j(x) = \sum_{k=0}^{m_j-4} \frac{(\tau_2 - a)^{m_j-3-k}}{(m_j-3-k)!} \frac{(\tau_3-\tau_2)^k}{k!}\, w_5 + \frac{(\tau_3-\tau_2)^{m_j-3}}{(m_j-3)!} x.$$

Then, the system (A) has (at least) three fixed-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ satisfying (3.26).

Proof. To apply Theorem 2.2, we shall define the following functionals on C :

$$\begin{aligned} &\gamma(u) = \|u\|, \qquad \beta(u) = \max_{1\le i\le n}\max_{t\in[\tau_1,\tau_4]} \theta_i u_i^{(m_i-3)}(t), \qquad \Theta(u) = \max_{1\le i\le n}\max_{t\in[\tau_2,\tau_3]} \theta_i u_i^{(m_i-3)}(t),\\ &\psi(u) = \min_{1\le i\le n}\min_{t\in[\tau_1,\tau_4]} \theta_i u_i^{(m_i-3)}(t), \qquad \alpha(u) = \min_{1\le i\le n}\min_{t\in[\tau_2,\tau_3]} \theta_i u_i^{(m_i-3)}(t). \end{aligned} \tag{3.39}$$

Using a similar argument as in the proof of Theorem 3.2, we can show that $S : \overline{P}(\gamma, w_5) \to \overline{P}(\gamma, w_5)$.

Next, we shall check that condition (a) of Theorem 2.2 is satisfied. As in the proof of Theorem 3.2, we see that $\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\} \ne \emptyset$. Let $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$. Then, we have (3.28) and also from $\Theta(u) \le w_4$,

$$\theta_j u_j^{(m_j-3)}(s) \le w_4, \quad s \in [\tau_2, \tau_3],\ 1 \le j \le n.$$

Integrating the above from τ_2 to s repeatedly and then applying (3.8) yields

$$\begin{aligned} \theta_j u_j(s) &\le \sum_{k=0}^{m_j-4} \theta_j u_j^{(k)}(\tau_2) \frac{(s-\tau_2)^k}{k!} + \frac{(s-\tau_2)^{m_j-3}}{(m_j-3)!} w_4 \\ &\le \sum_{k=0}^{m_j-4} \frac{(\tau_2-a)^{m_j-3-k}}{(m_j-3-k)!} \frac{(\tau_3-\tau_2)^k}{k!} w_5 + \frac{(\tau_3-\tau_2)^{m_j-3}}{(m_j-3)!} w_4 = \overline{\ell}_j(w_4), \quad s \in [\tau_2, \tau_3],\ 1 \le j \le n. \end{aligned} \tag{3.40}$$

Combining (3.28) and (3.40), we get

$$|u_j(s)| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!}\, w_3, \overline{\ell}_j(w_4)\right], \quad s \in [\max\{\tau_2, t^*\},\ \tau_3],\ 1 \le j \le n$$

or equivalently

$$|u_j(\phi_j(s))| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!} w_3, \overline{\ell}_j(w_4)\right], \quad s \in F,\ 1 \le j \le n. \tag{3.41}$$

An application of (3.15), (C10), (3.41) and (R″) gives

$$\begin{aligned} \alpha(Su) &\ge \min_{1\le i\le n}\min_{t\in[\tau_2,\tau_3]} \int_F g(t,s)\mu_i(s) p(u(\phi(s)))\mathrm{d}s \\ &> \min_{1\le i\le n}\min_{t\in[\tau_2,\tau_3]} \int_F g(t,s)\mu_i(s) \frac{w_3}{d_{4,i}}\mathrm{d}s = \min_{1\le i\le n} \frac{d_{4,i}}{d_{4,i}} w_3 = w_3. \end{aligned}$$

Hence, $\alpha(Su) > w_3$ for all $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$.

Now, to show that condition (b) of Theorem 2.2 is satisfied, we see, as in the proof of Theorem 3.2, that $\{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\} \ne \emptyset$. Let $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$. Then, we have (3.31) and (3.33).

Moreover, $\psi(u) \geq w_1$ implies $|u_j|_0 \geq w_1,\ 1 \leq j \leq n$, and so on using (3.12) we get

$$\theta_j u_j(s) \geq \frac{Mh^{m_j-3}}{(m_j-3)!}\, w_1, \quad s \in [t^*, t^*+h],\ 1 \leq j \leq n. \tag{3.42}$$

Coupling (3.31) and (3.42) yields

$$|u_j(s)| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!}\, w_1, \ell_j(w_2)\right], \quad s \in [\max\{\tau_1, t^*\},\ \tau_4],\ 1 \leq j \leq n$$

or equivalently

$$|u_j(\phi_j(s))| \in \left[\frac{Mh^{m_j-3}}{(m_j-3)!} w_1, \ell_j(w_2)\right], \quad s \in G,\ 1 \leq j \leq n. \tag{3.43}$$

Note that $\tau_4 > \max\{\tau_1, t^*\}$ since $\tau_4 > \tau_1$ and $\tau_4 \geq \tau_3 > \max\{\tau_2, t^*\} \geq t^*$. Using (3.15), (3.6), (C11), (3.33), (3.43), (P″) and (Q″), we find

$$\begin{aligned}
\beta(Su) &\leq \int_a^b g(t^*, s)\nu(s)p(u(\phi(s)))\mathrm{d}s \\
&= \int_G g(t^*, s)\nu(s)p(u(\phi(s)))\mathrm{d}s + \int_{[a,b]\setminus G} g(t^*, s)\nu(s)p(u(\phi(s)))\mathrm{d}s \\
&< \int_G g(t^*, s)\nu(s)\frac{1}{d_5}\left(w_2 - \frac{w_5 d_6}{q}\right)\mathrm{d}s + \int_{[a,b]\setminus G} g(t^*, s)\nu(s)\frac{w_5}{q}\mathrm{d}s \\
&= d_5\frac{1}{d_5}\left(w_2 - \frac{w_5 d_6}{q}\right) + d_6\frac{w_5}{q} = w_2.
\end{aligned}$$

Thus, $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$.

Next, we shall prove that condition (c) of Theorem 2.2 is fulfilled. In view of (3.15) and (3.6), we obtain for $u \in C$,

$$\Theta(Su) = \max_{1\leq i\leq n}\ \max_{t\in[\tau_2,\tau_3]} \theta_i(S_i u)^{(m_i-3)}(t) \leq \int_a^b g(t^*, s)\nu(s)p(u(\phi(s)))\mathrm{d}s. \tag{3.44}$$

Moreover, we also have (3.35), which together with (3.44), yields (3.36). The rest then follows as in the proof of Theorem 3.2.

Finally, to show that condition (d) of Theorem 2.2 is satisfied, using (3.15) and (3.6), we get for $u \in C$,

$$\beta(Su) = \max_{1\leq i\leq n}\ \max_{t\in[\tau_1,\tau_4]} \theta_i(S_i u)^{(m_i-3)}(t) \leq \int_a^b g(t^*, s)\nu(s)p(u(\phi(s)))\mathrm{d}s. \tag{3.45}$$

On the other hand, in view of (3.15), (3.7) and (C5), we obtain for $u \in C$,

$$\psi(Su) = \min_{1\leq i\leq n}\ \min_{t\in[\tau_1,\tau_4]} \theta_i(S_i u)^{(m_i-3)}(t) \geq \min_{1\leq i\leq n} \int_a^b Mg(t^*, s)\rho_i\nu(s)p(u(\phi(s)))\mathrm{d}s \geq \min_{1\leq i\leq n} M\rho_i \cdot \beta(Su) \tag{3.46}$$

where we have used (3.45) in the last inequality. Let $u \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Su) < w_1$. Then, it follows from (3.46) that

$$\beta(Su) \leq \frac{1}{\min\limits_{1\leq i\leq n} M\rho_i}\, \psi(Su) < \frac{1}{\min\limits_{1\leq i\leq n} M\rho_i}\, w_1 \leq \frac{1}{\min\limits_{1\leq i\leq n} M\rho_i}\, w_2 \cdot \min_{1\leq i\leq n} M\rho_i = w_2.$$

Hence, $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Su) < w_1$.

The conclusion is now immediate from Theorem 2.2. □

4. Examples

In this section we shall provide examples to illustrate the usefulness of the results obtained in Section 3.

Example 4.1. Consider the system (A) with

$$\begin{aligned} & n=2, \qquad m_1=3, \qquad m_2=4, \qquad [a,b]=[0,1], \qquad \phi_1(t)=\cos\frac{\pi t}{2}, \qquad \phi_2(t)=\sin\frac{\pi t}{2}, \\ & t^*=0.7, \qquad \xi=1, \qquad \delta=0.5, \end{aligned} \tag{4.1}$$

$$\begin{aligned} f_1(t,u_1,u_2) &= f_2(t,u_1,u_2)=p(u_1,u_2) \\ &= \begin{cases} \dfrac{w_1}{2q}, & (u_1,u_2)\in[0,w_1]\times[0,w_1]\equiv H_1 \\ \dfrac{1}{2}\left(\dfrac{d}{q}+\dfrac{w_2}{\min\{r_1,r_2\}}\right), & (u_1,u_2)\in\left[\dfrac{48}{49}w_2,\infty\right)\times\left[\dfrac{48}{490}w_2,\infty\right)\equiv H_2 \\ T(u_1,u_2), & (u_1,u_2)\in\mathbb{R}^2\setminus(H_1\cup H_2) \end{cases} \end{aligned} \tag{4.2}$$

where $T(u_1,u_2)$ is continuous in each argument and satisfies

$$\begin{cases} T(0,u_2)=T(w_1,u_2)=T(u_1,0)=T(u_1,w_1)=\dfrac{w_1}{2q}, \quad u_1,u_2\in[0,w_1]; \\ T\left(\dfrac{48}{49}w_2,u_2\right)=T\left(u_1,\dfrac{48}{490}w_2\right)=\dfrac{1}{2}\left(\dfrac{d}{q}+\dfrac{w_2}{\min\{r_1,r_2\}}\right), \\ \quad u_1\in\left[\dfrac{48}{49}w_2,\infty\right),\ u_2\in\left[\dfrac{48}{490}w_2,\infty\right); \\ 0\le T(u_1,u_2)\le\dfrac{1}{2}\left(\dfrac{d}{q}+\dfrac{w_2}{\min\{r_1,r_2\}}\right), \quad (u_1,u_2)\in\mathbb{R}^2\setminus(H_1\cup H_2); \end{cases} \tag{4.3}$$

and w_i's and d are as in the context of Theorem 3.1 satisfying

$$0<w_1<w_2<\frac{w_2}{M\ \min\{\rho_1,\rho_2\}}\le w_3\le d, \qquad w_1<\frac{48}{490}\,w_2, \qquad d>\frac{qw_2}{\min\{r_1,r_2\}}. \tag{4.4}$$

We are interested in the *positive* solutions of (A) with (4.1)–(4.4), so we let $\theta_1=\theta_2=1$. Fix $h=0.1$ and the functions $\mu_1=\mu_2=\nu\equiv 1$ (this implies $\rho_1=\rho_2=1$). Using the explicit expression of $g(t,s)$ in Lemma 3.1, we find that

$$\begin{aligned} & D=\left[\max\left\{\frac{2}{\pi}\cos^{-1}0.8,\ \frac{2}{\pi}\sin^{-1}0.7\right\},\ \min\left\{\frac{2}{\pi}\cos^{-1}0.7,\ \frac{2}{\pi}\sin^{-1}0.8\right\}\right]=[0.4936,0.5064], \\ & M=\frac{48}{49}, \qquad q=\int_0^1 g(0.7,s)\mathrm{d}s=0.2291, \qquad r_1=r_2=\min_{t\in[0.6,0.8]}\int_D g(t,s)\mathrm{d}s=0.002880. \end{aligned}$$

Hence, (4.4) reduces to

$$0<\frac{490}{48}\,w_1<w_2<\frac{49}{48}\,w_2\le w_3\le d \quad \text{and} \quad d>79.55w_2. \tag{4.5}$$

We shall check the conditions of Theorem 3.1. Clearly, (C1)–(C7) and condition (P) are satisfied. Next, from (4.4) we have $\frac{w_2}{\min\{r_1,r_2\}}<\frac{d}{q}$, therefore we get for $(u_1,u_2)\in[0,d]\times[0,d]$,

$$p(u_1,u_2)\le\frac{1}{2}\left(\frac{d}{q}+\frac{w_2}{\min\{r_1,r_2\}}\right)<\frac{1}{2}\left(\frac{d}{q}+\frac{d}{q}\right)=\frac{d}{q}.$$

Hence, condition (Q2) is fulfilled. Finally, (R) is satisfied since for $(u_1,u_2)\in[Mw_2,w_3]\times[Mhw_2,w_3]=\left[\frac{48}{49}\,w_2,\ w_3\right]\times\left[\frac{48}{490}\,w_2,\ w_3\right]$ we have

$$p(u_1,u_2)=\frac{1}{2}\left(\frac{d}{q}+\frac{w_2}{\min\{r_1,r_2\}}\right)>\frac{1}{2}\left(\frac{w_2}{\min\{r_1,r_2\}}+\frac{w_2}{\min\{r_1,r_2\}}\right)=\frac{w_2}{\min\{r_1,r_2\}}.$$

By Theorem 3.1, the system (A) with (4.1)–(4.3) and (4.5) has (at least) three *positive* solutions $u^1, u^2, u^3 \in C$ such that

$$\begin{aligned} &\|u^1\| < w_1; \qquad u_1^2(t), (u_2^2)'(t) > w_2, \quad t \in [0.6, 0.8]; \\ &\|u^3\| > w_1 \quad \text{and} \quad \min\left\{\min_{t\in[0.6,0.8]} u_1^3(t), \min_{t\in[0.6,0.8]} (u_2^3)'(t)\right\} < w_2. \end{aligned} \tag{4.6}$$

Example 4.2. Consider the system (A) with

$$\begin{aligned} &n = 2, \quad m_1 = 3, \quad m_2 = 4, \quad [a, b] = [0, 1], \quad \phi_1(t) = t^2, \quad \phi_2(t) = t^3, \\ &t^* = 0.7, \quad \xi = 1, \quad \delta = 0.5, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &f_1(t, u_1, u_2) = f_2(t, u_1, u_2) = p(u_1, u_2) \\ &= \begin{cases} \dfrac{1}{2d_2}\left(w_2 - \dfrac{w_5 d_3}{q}\right), & (u_1, u_2) \in [0, w_2] \times [0, 0.99w_2] \equiv H_1 \\ \dfrac{1}{2}\left(\dfrac{w_3}{\min\{r_1, r_2\}} + \dfrac{w_3}{\min\{d_{1,1}, d_{1,2}\}}\right), & (u_1, u_2) \in [0.8724w_3, \infty) \times [0.2181w_3, \infty) \equiv H_2 \\ T(u_1, u_2), & (u_1, u_2) \in \mathbb{R}^2 \setminus (H_1 \cup H_2) \end{cases} \end{aligned} \tag{4.8}$$

where $T(u_1, u_2)$ is continuous in each argument and satisfies

$$\begin{cases} T(0, u_2) = T(w_2, u_2) = T(u_1, 0) = T(u_1, 0.99w_2) = \dfrac{1}{2d_2}\left(w_2 - \dfrac{w_5 d_3}{q}\right), \\ \quad u_1 \in [0, w_2], u_2 \in [0, 0.99w_2]; \\ T(0.8724w_3, u_2) = T(u_1, 0.2181w_3) = \dfrac{1}{2}\left(\dfrac{w_3}{\min\{r_1, r_2\}} + \dfrac{w_3}{\min\{d_{1,1}, d_{1,2}\}}\right), \\ \quad u_1 \in [0.8724w_3, \infty), u_2 \in [0.2181w_3, \infty); \\ 0 \le T(u_1, u_2) \le \dfrac{1}{2}\left(\dfrac{w_3}{\min\{r_1, r_2\}} + \dfrac{w_3}{\min\{d_{1,1}, d_{1,2}\}}\right), \quad (u_1, u_2) \in \mathbb{R}^2 \setminus (H_1 \cup H_2); \end{cases} \tag{4.9}$$

and w_i's are as in the context of Theorem 3.2 satisfying

$$0 < w_2 < w_3 < \frac{w_3}{M \ \min\{\rho_1, \rho_2\}} \le w_4 \le w_5 < \frac{q w_2}{d_3}, \quad 0.99w_2 < 0.2181w_3, \quad w_5 > \frac{q w_3}{\min\{r_1, r_2\}}. \tag{4.10}$$

We are interested in the *positive* solutions of (A) with (4.7)–(4.10), therefore we let $\theta_1 = \theta_2 = 1$. Fix $h = 0.25$, the functions $\mu_1 = \mu_2 = \nu \equiv 1$ (this implies $\rho_1 = \rho_2 = 1$) and

$$\tau_1 = 0 < t^* - h < \tau_2 = 0.6 < \tau_3 = 0.8 < t^* + h < \tau_4 = 0.99.$$

By direct computation, we get

$$\begin{aligned} &D = \left[\max\{(0.7)^{\frac{1}{2}}, (0.7)^{\frac{1}{3}}\}, \min\{(0.95)^{\frac{1}{2}}, (0.95)^{\frac{1}{3}}\}\right] = [0.8879, 0.9747], \\ &E = \left[0, \min\{(0.99)^{\frac{1}{2}}, (0.99)^{\frac{1}{3}}\}\right] = [0, 0.9950], \qquad M = \frac{171}{196}, \qquad q = \int_0^1 g(0.7, s)\mathrm{d}s = 0.2291, \\ &r_1 = r_2 = \min_{t\in[0.45,0.95]} \int_D g(t, s)\mathrm{d}s = 0.03108, \qquad d_{1,1} = d_{1,2} = \min_{t\in[0.6,0.8]} \int_D g(t, s)\mathrm{d}s = 0.03490, \\ &d_2 = \int_E g(0.7, s)\mathrm{d}s = 0.2280, \qquad d_3 = q - d_2 = 0.001100. \end{aligned}$$

Hence, (4.10) reduces to

$$0 < 4.539w_2 < w_3 < \frac{196}{171} w_3 \le w_4 \le w_5 < 208.3w_2 \quad \text{and} \quad w_5 > 7.371w_3. \tag{4.11}$$

We shall check the conditions of Theorem 3.2. It is clear that (C1)–(C5), (C8), (C9) and condition (P′) are fulfilled (note that $\ell_1(w_2) = w_2,\ \ell_2(w_2) = 0.99w_2$). Next, since

$$\min\{r_1, r_2\} < \min\{d_{1,1}, d_{1,2}\} < d_2 \quad \text{and} \quad \frac{w_3}{\min\{r_1, r_2\}} < \frac{w_5}{q} \quad (\text{from (4.10)}), \tag{4.12}$$

we find for $(u_1, u_2) \in [0, w_5] \times [0, w_5]$,

$$p(u_1, u_2) \le \frac{1}{2}\left(\frac{w_3}{\min\{r_1, r_2\}} + \frac{w_3}{\min\{d_{1,1}, d_{1,2}\}}\right) < \frac{1}{2}\left(\frac{w_3}{\min\{r_1, r_2\}} + \frac{w_3}{\min\{r_1, r_2\}}\right) = \frac{w_3}{\min\{r_1, r_2\}} < \frac{w_5}{q}.$$

Hence, condition (Q′) is met. Finally, (R′) is satisfied since for $(u_1, u_2) \in [Mw_3, \ell_1(w_4)] \times [Mhw_3, \ell_2(w_4)] = [0.8724w_3, w_4] \times [0.2181w_3, 0.99w_4]$, using (4.12) we get

$$p(u_1, u_2) = \frac{1}{2}\left(\frac{w_3}{\min\{r_1, r_2\}} + \frac{w_3}{\min\{d_{1,1}, d_{1,2}\}}\right) > \frac{1}{2}\left(\frac{w_3}{\min\{d_{1,1}, d_{1,2}\}} + \frac{w_3}{\min\{d_{1,1}, d_{1,2}\}}\right) = \frac{w_3}{\min\{d_{1,1}, d_{1,2}\}}.$$

It follows from Theorem 3.2 that the system (A) with (4.7)–(4.9) and (4.11) has (at least) three *positive* solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ such that

$$u_1^1(t), (u_2^1)'(t) < w_2, \quad t \in [0, 0.99]; \qquad u_1^2(t), (u_2^2)'(t) > w_3, \quad t \in [0.6, 0.8];$$

$$\max\left\{\max_{t\in[0,0.99]} u_1^3(t), \max_{t\in[0,0.99]} (u_2^3)'(t)\right\} > w_2 \quad \text{and} \quad \min\left\{\min_{t\in[0.6,0.8]} u_1^3(t), \min_{t\in[0.6,0.8]} (u_2^3)'(t)\right\} < w_3. \tag{4.13}$$

Remark 4.1. In Example 4.2, we see that for $(u_1, u_2) \in [Mw_3, w_4] \times [Mhw_3, w_4] = [0.8724w_3, w_4] \times [0.2181w_3, w_4]$,

$$p(u_1, u_2) = \frac{1}{2}\left(\frac{w_3}{\min\{r_1, r_2\}} + \frac{w_3}{\min\{d_{1,1}, d_{1,2}\}}\right) < \frac{1}{2}\left(\frac{w_3}{\min\{r_1, r_2\}} + \frac{w_3}{\min\{r_1, r_2\}}\right) = \frac{w_3}{\min\{r_1, r_2\}}.$$

Thus, condition (R) of Theorem 3.1 is *not* satisfied. Example 4.2 illustrates the case when Theorem 3.2 is applicable but not Theorem 3.1. Hence, this example shows that Theorem 3.2 is indeed more general than Theorem 3.1.

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